On some applications of an integral formula of Hurwitz

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Hohto Bekki

Keio University

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§0. Introduction

Eisenstein series: For $k \geq 1$,

$$G_{2+2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2, \neq 0} \frac{1}{(m+nz)^{2+2k}}, \quad z \in \mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Hecke's formula (example): Let $w \in \mathfrak{h}$.

$$\int_w^{\frac{2w+1}{w+1}} (-z^2+z+1)^k G_{2+2k}(z)dz$$

$$= \begin{cases} \dfrac{2k!5^{k-1}\sqrt{5}}{(2k+1)!}\zeta_{\mathbb{Q}(\sqrt{5})}(k+1) & \text{if } k+1\text{: even} \\ 0 & \text{if } k+1\text{: odd.} \end{cases}$$

Siegel-Klingen (example): $\zeta_{\mathbb{Q}(\sqrt{5})}(2) = \dfrac{2\pi^4}{75\sqrt{5}}$.

Remark: There are many (cohomological) generalizations of such a story for critical values of totally real fields or CM fields. (Eisenstein cocycles)

Questions:

- For $k+1$: odd?
- For more general number fields?

Today: Report on

§1 A cohomological generalization of Hecke's formula for general number fields. (arXiv:2104.09030)

§2 An observation towards applications: a relation between values of zeta functions of totally real fields and the conical zeta values.

Both of the topics are based on a classical formula of Hurwitz.

Key (Hurwitz's formula): First recall

$$\int_a^b \frac{dz}{(m+nz)^2} = \frac{b-a}{(m+na)(m+nb)},$$

and for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$,

$$\int_a^b \left(\frac{b-z}{b-a}\right)^{k_1} \left(\frac{z-a}{b-a}\right)^{k_2} \frac{dz}{(m+nz)^{2+k_1+k_2}} = \frac{k_1!k_2!}{(k_1+k_2+1)!} \frac{b-a}{(m+na)^{k_1+1}(m+nb)^{k_2+1}}.$$

$\rightsquigarrow$ Hecke's formula

A generalization of this formula is used by Hurwitz (1922), which is also known as the Feynman parametrization.

Hurwitz's formula: Let

- $x \in \mathbb{C}^g - \{0\}$,
- $\xi_1, \ldots, \xi_g \in \mathbb{C}^g$: basis such that $\mathrm{Re}\langle x, \xi_i \rangle > 0$ for all i, where $\langle x, y \rangle$ is the usual scalar product of x and y,
- $\xi_1^*, \ldots, \xi_g^* \in \mathbb{C}^g$: the dual basis with respect to $\langle x, y \rangle$,
- $\sigma(\xi_1, \ldots, \xi_g) := \left\{ \sum_{i=1}^{g} t_i \xi_i \in \mathbb{C}^g \;\middle|\; \sum_{i=1}^{g} t_i = 1, t_i \geq 0 \right\}$,
- $\omega(y) := \sum_{i=1}^{g} (-1)^{i-1} y_i dy_1 \wedge \cdots \wedge \check{dy_i} \wedge \cdots \wedge dy_g$: $(g-1)$-form on $\mathbb{C}^g$,
- $\boldsymbol{k} = (k_1, \ldots, k_g) \in (\mathbb{Z}_{\geq 0})^g$, $|\boldsymbol{k}| = k_1 + \cdots + k_g$, $\boldsymbol{k}! = k_1! \cdots k_g!$.

Then we have

$$\int_{\sigma(\xi_1, \ldots, \xi_g)} \langle \xi_1^*, y \rangle^{k_1} \cdots \langle \xi_g^*, y \rangle^{k_g} \frac{\omega(y)}{\langle x, y \rangle^{g+|\boldsymbol{k}|}} = \frac{\boldsymbol{k}!}{(g+|\boldsymbol{k}|-1)!} \frac{\det(\xi_1 \cdots \xi_g)}{\langle x, \xi_1 \rangle^{k_1+1} \cdots \langle x, \xi_g \rangle^{k_g+1}}.$$

§1. Cohomological generalization of Hecke's formula

⋆ There are many theories for critical values of totally real or CM fields, e.g. Harder, Sczech, Nori, Solomon, Hill, Vlasenko-Zagier, Beilinson-Kings-Levin, Charollois-Dasgupta-Greenberg, Bergeron-Charollois-Garcia, Bannai-Hagihara-Yamada-Yamamoto, etc.

$\rightsquigarrow$ general number fields?

"Eisenstein series": $g, k \in \mathbb{Z}_{\geq 1}$. Consider (formally)

$$\psi_{kg}(y) = \sum_{x \in \mathbb{Z}^g, \neq 0} \frac{1}{\langle x, y \rangle^{g+kg}}, \quad y \in \mathbb{C}^g - \{0\}.$$

- $g = 2$, $y = (1, z)$, $z \in \mathfrak{h} \Longrightarrow \psi_{2k}(y) = G_{2+2k}(z)$.
- $g \geq 3 \Longrightarrow$ not convergent!

⋆ Justify ψ_{kg} as an equivariant cohomology class.

Shintani's method:

$\star$ We combine some ideas and techniques developed by Vlasenko-Zagier, Charollois-Dasgupta-Greenberg, Bannai-Hagihara-Yamada-Yamamoto.

- Decomposition along cones: For $C \subset \mathbb{R}^g$: rational cone, i.e., $C = \sum_{i=1}^{g} \mathbb{R}_{>0}\alpha_i$ with $\alpha_i \in \mathbb{Q}^g$, consider

$$\psi_{kg,C}(y) := \sum_{x \in \mathbb{Z}^g \cap C} \frac{1}{\langle x, y \rangle^{g+kg}} \rightsquigarrow \text{ converges on some region.}$$

Then by putting these fragments together,

$$\Psi_{kg} = (\psi_{kg,C})_C \in H^{g-1}(Y^\circ, SL_g(\mathbb{Z}), \mathscr{F}^{\Xi}_{kg}),$$

Cohomological realization of ψ_{kg}: Shintani-Barnes cocycle (class)

where $\begin{cases} Y^\circ := \mathbb{C}^g - \sqrt{-1}\mathbb{R}^g \curvearrowleft SL_g(\mathbb{Z}) \\ \mathscr{F}^{\Xi}_{kg}\text{: a certain } SL_g(\mathbb{Z})\text{-equivariant sheaf.} \end{cases}$

Specialization to zeta values: For

- $F/\mathbb{Q}$: number field of degree g,
- $\mathfrak{a} \subset F$: a fractional ideal,
- $k \in \mathbb{Z}_{\geq 1}$,

$\exists$ a natural specialization map

$$\begin{array}{ccc} H^{g-1}(Y^{\circ}, SL_g(\mathbb{Z}), \mathscr{F}_{kg}^{\Xi}) & \xrightarrow{\text{“integration”}} & \mathbb{C} \\ \Cup & & \Cup \\ \Psi_{kg} & \longmapsto & c_{\mathfrak{a},k+1}\zeta_F(\mathfrak{a}^{-1}, k+1), \end{array}$$

where $\begin{cases} c_{\mathfrak{a},k+1} = \dfrac{\sqrt{d_F}(k!)^g N\mathfrak{a}}{(g+gk-1)!} \text{ with } d_F\text{: discriminant of } F, \\ \zeta_F(\mathfrak{a}^{-1}, s)\text{: the partial zeta function associated to } \mathfrak{a}^{-1}. \end{cases}$

This can be proved by using the Hurwitz's formula.

$\star$ We may symbolically write this fact as

$$\int N_F^k \Psi_{kg} \omega = c_{\mathfrak{a},k+1} \zeta_F(\mathfrak{a}^{-1}, k+1),$$

since the above specialization map is actually defined by the multiplication by the norm N_F^k and the integration over a certain $(g-1)$-simplex in $\mathbb{C}^g$.

: Cohomological generalization of Hecke's formula

I am now studying the applications.

<u>An observation</u>:
If F is totally real, then the left hand side $\int N_F^k \Psi_{kg} \omega$ is related to a kind of conical zeta values for non-rational cones.

§2. Conical zeta values

Conical zeta values: For

- a cone $C = \sum_{i=1}^{g} \mathbb{R}_{>0}\alpha_i \subset \mathbb{R}^g_{\geq 0}$ with $\alpha_i \in \mathbb{R}^g_{\geq 0}$,
- $v \in \mathbb{Q}^g$,
- $\boldsymbol{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g_{\geq 1}$,

define

$$\zeta_C(\boldsymbol{k}, v) := \sum_{x \in (v+\mathbb{Z}^g) \cap C} \frac{1}{x^{\boldsymbol{k}}},$$

whenever the series converges, where $x^{\boldsymbol{k}} = x_1^{k_1} \cdots x_g^{k_g}$ as usual.
If $v = 0$, simply put

$$\zeta_C(\boldsymbol{k}) := \zeta_C(\boldsymbol{k}, 0).$$

Remarks: Let $e_1, \dots, e_g \in \mathbb{R}^g$: standard basis, i.e., $e_i = (0, \dots, \overset{i}{1}, 0, \dots)$.

1. If $C = \sum_{i=1}^{g} \mathbb{R}_{>0} e_i = \mathbb{R}^g_{>0}$, then
$$\zeta_C(\boldsymbol{k}) = \zeta(k_1) \cdots \zeta(k_g).$$
2. If $C = \mathbb{R}_{>0} e_1 + \mathbb{R}_{>0}(e_1 + e_2) + \cdots + \mathbb{R}_{>0}(e_1 + \cdots + e_g)$, then
$$\zeta_C(\boldsymbol{k}) = \zeta(k_g, k_{g-1}, \dots, k_1)\text{: MZV.}$$
3. More generally, if $\alpha_i \in \mathbb{Q}^g$, i.e., C is a rational cone, then it is known
$$\zeta_C(\boldsymbol{k}) \in \langle \text{cyclotomic MZV} \rangle_{\mathbb{Q}},$$
and the relations among them are studied. (Terasoma, Panzer, Guo-Paycha-Zhang)

Questions:

- What can we say about $\zeta_C(\boldsymbol{k})$ for non-rational cones C?
- Are they studied?

Relation to Dedekind zeta values (example): Consider

$$C = \mathbb{R}_{>0} \begin{pmatrix} 1 \\ \frac{3+\sqrt{5}}{2} \end{pmatrix} + \mathbb{R}_{>0} \begin{pmatrix} 1 \\ \frac{3-\sqrt{5}}{2} \end{pmatrix}.$$

Then we have the following.

Proposition (known?)

For all integer $k \geq 2$, we have

$$\zeta_{\mathbb{Q}(\sqrt{5})}(k) \in \frac{1}{\sqrt{5}} \sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 = 2k}} \mathbb{Q}\ \zeta_C(k_1, k_2).$$

Moreover, the coefficients can be computed explicitly.

e.g. We find

$$\begin{aligned}\zeta_{\mathbb{Q}(\sqrt{5})}(2) &= \frac{1}{5\sqrt{5}}\left(4\zeta_C(3,1)+3\zeta_C(2,2)\right)\\ &= 1.1616711\cdots\\ \Big(&= \frac{2\pi^4}{75\sqrt{5}}\Big)\end{aligned}$$

$$\begin{aligned}\zeta_{\mathbb{Q}(\sqrt{5})}(3) &= \frac{1}{25\sqrt{5}}\left(12\zeta_C(5,1)+18\zeta_C(4,2)+11\zeta_C(3,3)\right)\\ &= 1.0275480\cdots\end{aligned}$$

Totally real fields: More generally, let

- $F/\mathbb{Q}$: totally real field of degree g,
- $E \subset \mathbb{R}$: Galois closure of F.

A cone $C = \sum_{i=1} \mathbb{R}_{>0}\alpha_i$ is called E-rational if we can take $\alpha_i \in E^g$.

Then we have the following.

Proposition (known?)

For all integer $k \geq 2$ and a fractional ideal $\mathfrak{a} \subset F$, there exist

- $C_1, \ldots, C_r$*: E-rational cones and*
- $v_1, \ldots, v_r \in \mathbb{Q}^g$

such that

$$\zeta_F(\mathfrak{a}^{-1}, k) \in \frac{1}{\sqrt{d_F}} \sum_{i=1}^{r} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_{\geq 1}^g \\ |\boldsymbol{k}| = kg}} \mathbb{Q}\zeta_{C_i}(\boldsymbol{k}, v_i).$$

This can be also proved very easily by using the Hurwitz's formula.

Remarks and Questions:

- Are these conical zeta values for non-rational (algebraic) cones studied?
- Are these formulas known? (possibly in another form?)
- Can we say anything about the individual values $\zeta_C(\boldsymbol{k}, v)$? e.g. When do they become periods?
- It doesn't seem (a priori) that there are a nice "algebraic" (iterated) integral representation for $\zeta_C(\boldsymbol{k}, v)$ if C is not rational.
- Are there any efficient ways to compute the values $\zeta_C(\boldsymbol{k}, v)$?

Any comments or information would be greatly appreciated!!

Thank you very much for your attention!!