2024/6/11 ISoAF.

On the denominators of the special values of the partial zeta functions of real quadratic fields.

joint w/ Ryotaro Sakamoto.

§1. Intro & Main Results

F : real quad. fld. eg. $\mathbb{Q}(\sqrt{5})$

$\mathcal{O} \subset F$: order eg. $\mathcal{O} = \mathcal{O}_F$

$Cl_{\mathcal{O}}^{+}$: the narrow ideal class gp of $\mathcal{O}$

$\mathcal{A} \in Cl_{\mathcal{O}}^{+}$

$$\rightsquigarrow \zeta_{\mathcal{O}}(\mathcal{A}, s) := \sum_{\substack{\mathfrak{a} \in \mathcal{A} \\ \mathfrak{a} \subset \mathcal{O}}} \frac{1}{N\mathfrak{a}^s} \quad : \mathrm{Re}(s) > 1 \rightsquigarrow s \in \mathbb{C} \text{ merom.}$$

: partial zeta function.

$\zeta(-1) = -\frac{1}{12}$,

eg. $\zeta_{\mathbb{Q}(\sqrt{5})}(-1) = \frac{1}{30}$

Known: (also for tot real flds.)

- (Rationality) $\forall k \in \mathbb{Z}_{\geq 2}$, $\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Q}$. (Klingen-Siegel)

- (Integrality / p-adic interpolation)

  $\mathcal{O} = \mathcal{O}_F$ (for safety), $\mathfrak{c} \subset \mathcal{O}_F$: int ideal.

  $$\rightsquigarrow \zeta_{\mathcal{O}_F, \mathfrak{c}}(\mathcal{A}, 1-k) := \zeta_{\mathcal{O}_F}(\mathcal{A}, 1-k) - N\mathfrak{c}^k \zeta_{\mathcal{O}_F}(\mathfrak{c}\mathcal{A}, 1-k) \in \mathbb{Z}\left[\tfrac{1}{N\mathfrak{c}}\right]$$

  $\rightsquigarrow$ p-adic interpolation of $\zeta_{\mathcal{O}_F, \mathfrak{c}}(\mathcal{A}, 1-k)$

  (Coates-Sinnott, Deligne-Ribet)

Today: The universal upper bound for denom $(\zeta_{\mathcal{O}}(A, 1-k))$

$\xrightarrow{\text{application}}$ Harder's thry on denom of Eis class.

For $k \in \mathbb{Z}_{\geq 2}$.

$$\mathcal{D}_k := \left\{ J \in \mathbb{Z}_{>0} \;\middle|\; \begin{array}{l} \forall F : \text{real quad. } \forall \mathcal{O} \subset F, \forall A \in Cl_{\mathcal{O}}^{+} \\ J\, \zeta_{\mathcal{O}}(A, 1-k) \in \mathbb{Z} \end{array} \right\}$$

: universal upper bounds for denom $(\zeta_{\mathcal{O}}(A, 1-k))$.

Q. Can we determine $\mathcal{D}_k$??

- Zagier's observation.

Thm (Zagier 1977)

$$\zeta_{\mathcal{O}}(A, 1-k) = \sum_{i=1}^{\exists n} \sum_{j=1}^{2k-2} d_{kj}^{(i)} \left(\frac{B_{2k}}{2k} \frac{a_i^{2k-1-j}}{2k-1-j} - \frac{B_{j+1}}{j+1} \cdot \frac{B_{2k-1-j}}{2k-1-j} \right)$$

where

- $d_{kj}^{(i)}$, $a_i \in \mathbb{Z}$: def'd by continued fraction
- B_j : j-th Bernoulli # $\in \mathbb{Q}$.

Cor. $\operatorname{lcm}\left\{ \operatorname{denom}\left(\frac{B_{2k}}{2k} \frac{1}{2k-1-j} \right), \operatorname{denom}\left(\frac{B_{j+1}}{j+1} \cdot \frac{B_{2k-1-j}}{2k-1-j} \right) \right\} \in \mathcal{D}_k$.

eg. $k=2$, $\operatorname{lcm} = 720 \in \mathcal{D}_2$ ($\leftrightarrow$ $120 \in \mathcal{D}_2$)

$k=3$, $\operatorname{lcm} = 15120 \in \mathcal{D}_3$ ($\leftrightarrow$ $252 \in \mathcal{D}_3$)

- **Duke's conj.** Set $\zeta(1-2k) =: \pm \frac{N_{2k}}{J_{2k}}$ ← num. ← denom.

eg $\zeta(-3) = \frac{1}{120}$, $\zeta(-5) = -\frac{1}{252}$, ... $\zeta(-11) = \frac{691}{32760}$, ...

Conj (Duke 2022)

| $J_{2k} \in \mathcal{D}_k$ i.e. $J_{2k}\,\zeta_\Theta(\mathcal{A}, 1-k) \in \mathbb{Z}$, $\forall F, \forall \Theta, \forall \mathcal{A}$.

Main Result

Thm. Duke's conj is true & J_{2k} is best possible

| i.e. $\mathcal{D}_k = \mathbb{Z}_{>0} \cdot J_{2k}$.

Rmk Duke's conj is proved also by O'Sullivan.

§2. Strategy of pf.

Key: • integral repn of $\zeta_\Theta(\mathcal{A}, 1-k)$ & its cohom. interpret.

• Harder's work on Eisenstein class for $SL_2(\mathbb{Z})$.

• $+\alpha$ (Hida thry).

$$E_{2k}(z) = 1 + \frac{2}{\zeta(1-2k)} \sum_{m=1}^{\infty} \sigma_{2k-1}(m)\, q^m, \quad q = e^{2\pi i z}.$$

= Eis series of wt $2k$,

Integral rep'n

eg. $F=\mathbb{Q}(\sqrt{5})$. $\mathcal{O}=\mathcal{O}_F=\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. $\mathcal{A}=\mathrm{id}$.

$\forall\, \tau\in\mathbb{H}=\{z\in\mathbb{C}\mid \mathrm{Im}(z)\}$

$\begin{pmatrix}3&-1\\1&\end{pmatrix}\cdot\tau \rightsquigarrow \frac{3\tau-1}{\tau}$

$$\underbrace{\int_\tau^{\frac{3\tau-1}{\tau}}(-z^2+3z-1)^{k-1}}_{\cap}\ \underbrace{E_{2k}(z)\,dz}_{\cap} = (-1)^k\,\frac{\zeta_{\mathcal{O}}(\mathcal{A},1-k)}{\zeta(1-2k)}$$

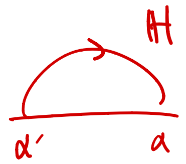
$$\cap$$

$$H_1(SL_2(\mathbb{Z})\backslash\mathbb{H}, M_{2k-2})\times H^1(Y, M_{2k-2}^\vee\otimes\mathbb{C})\xrightarrow{\langle,\rangle}\mathbb{C}$$

Γ $\overset{\cdot\cdot}{=}$ $SL_2(\mathbb{Z})\circlearrowright M_{2k-2}=\mathrm{Sym}^{2k-2}\mathbb{Z}^2$, $M_{2k-2}^\vee=\mathrm{Hom}_{\mathbb{Z}}(M_{2k-2},\mathbb{Z})$

$Y:=\Gamma\backslash\mathbb{H}$.

More prec.



$\rightsquigarrow \gamma$: hyp elt $\rightsquigarrow \left[\int_\tau^{\gamma\tau}(\text{"}\gamma\text{-inv. poly"})^{k-1}\right]$

$$\mathrm{Eis}_{2k-2}:=[E_{2k}(z)\,dz]\in H^1(Y, M_{2k-2}^\vee\otimes\mathbb{C})$$

$$\times$$

$\bigsqcup\limits_{\substack{F:\mathbb{RQ}.\\ \mathcal{O}\subset F}}\ Cl_{\mathcal{O}}^+\ni\mathcal{A}\longmapsto\ ^{\exists}\,\mathfrak{z}_{\mathcal{A},k}\in H_1(Y, M_{2k-2})$

$$\downarrow\langle,\rangle$$

$$\mathbb{C}$$

$$\langle \mathrm{Eis}_{2k-2},\ \mathfrak{z}_{\mathcal{A},k}\rangle=(-1)^k\frac{\zeta_{\mathcal{O}}(\mathcal{A},1-k)}{\zeta(1-2k)}=\pm\frac{J_{2k}\,\zeta_{\mathcal{O}}(\mathcal{A},1-k)}{N_{2k}}$$

§2.1 Harder's thm & Duke's conj.

Thm. (Harder, unpublished)

(0) (classical) $\mathrm{Eis}_{2k-2} \in H^1(Y, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{Q})$.

$\rightsquigarrow \mathrm{denom}(\mathrm{Eis}_{2k-2}) := \min\{\Delta \in \mathbb{Z}_{>0} \mid \Delta\, \mathrm{Eis}_{2k-2} \in H^1(Y, \mathcal{M}_{2k-2})_{/\mathrm{tor}}\}$

(1) $\mathrm{denom}(\mathrm{Eis}_{2k-2}) = \mathrm{numerator}(\zeta(1-2k)) = N_{2k}$.

<u>Rmk.</u> $\underset{\substack{q\text{-ser.} \\ (\text{de Rham})}}{\mathrm{denom}}(E_{2k}) \underset{\substack{\uparrow \\ \text{easy}}}{=} N_{2k} \underset{\substack{\uparrow \\ \text{hard}}}{=} \mathrm{denom}_{\mathrm{Betti}}(\mathrm{Eis}_{2k-2})$

<u>Cor.</u> Duke's conj holds true.

$$J_{2k}\zeta_{\mathcal{O}}(A, 1-k) = \pm\langle \overbrace{N_{2k}\mathrm{Eis}_{2k-2}}^{\text{int class}}, \overbrace{\mathfrak{z}_{A,k}}^{\text{int. class}} \rangle \in \mathbb{Z}. \qquad \square$$

<u>§ 2.2. $\min \mathcal{D}_k = J_{2k}$.</u>

<u>Summary.</u>

$$\begin{tikzcd}
\mathfrak{z}_{A,k} \in & H_1(Y, \mathcal{M}_{2k-2}) \arrow[r, "{\langle \mathrm{Eis}_{2k-2}, - \rangle}"] & \frac{1}{N_{2k}}\mathbb{Z} \subset \mathbb{C} \\
A \in \arrow[u, mapsto] \arrow[rr, bend right, mapsto] & \mathbb{Z}\Big[\bigsqcup_{\substack{F: \mathbb{R}\mathbb{Q} \\ \mathcal{O} \subset F}} \mathrm{Cl}_{\mathcal{O}}^+\Big] \arrow[u, "\mathfrak{z}_k"] \arrow[ur, "\varphi"'] & \pm \frac{1}{N_{2k}} \cdot J_{2k}\zeta_{\mathcal{O}}(A, 1-k). \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

<u>NB</u> $\min \mathcal{D}_k = J_{2k} \iff \varphi : \mathrm{surj}$.

• IF $\mathfrak{z}_k : \mathrm{surj} \Rightarrow$ done.

<u>Problem</u>: We couldn't prove this (unless $k=1$. $\to M_{2k-2}=\mathbb{Z}$)

<u>Solution</u>: prove im z_k generates suff. large subsp.

of $H^1(Y, M_{2k-2})\otimes\mathbb{Z}_p$, namely, "p-ordinary part".
$\circlearrowleft T_p$

<u>Outline</u>: Assume $p\geq 5$ for simplicity, $Y_1(p) := \Gamma_1(p)\backslash\mathbb{H}$.

Hida's control thm.

$$\begin{tikzcd}
H_1^{ord}(Y_1(p), \mathbb{Z}/p) \arrow[r, phantom, "\simeq"] & H_1^{ord}(Y_1(p), M_{2k-2}/p) & & \\
H_1^{ord}(Y_1(p), \mathbb{Z}_p) \arrow[u, "\mathrm{mod}\,p"] \arrow[r, phantom, "\text{``}\circlearrowright\text{''}"] & H_1^{ord}(Y_1(p), M_{2k-2}\otimes\mathbb{Z}_p) \arrow[u, "\mathrm{mod}\cdot p"] \arrow[r] \arrow[rr, bend left, two heads, "\langle Eis_{2k-2}, -\rangle"] & H_1^{ord}(Y, M_{2k-2}\otimes\mathbb{Z}_p) \arrow[r, two heads] & \frac{1}{N_{2k}}\mathbb{Z}_p \\
\mathbb{Z}_p[\bigsqcup_{F,\mathcal{O}} Cl_{\mathcal{O}}^+] \arrow[u, "z_1(\Gamma_1(p))"', "\text{surj}"] \arrow[r, phantom, "="] & \mathbb{Z}_p[\bigsqcup_{F,\mathcal{O}} Cl_{\mathcal{O}}^+] \arrow[u, "z_k(\Gamma_1(p))"] \arrow[r, two heads] & \mathbb{Z}_p[\bigsqcup_{F,\mathcal{O}} Cl_{\mathcal{O}}^+] \arrow[u, "z_k"] \arrow[ur, "\text{surj } \cancel{?}"] &
\end{tikzcd}$$

OK $\square$