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閉測地線から定まるモジュラー曲線上のホモロジー類について

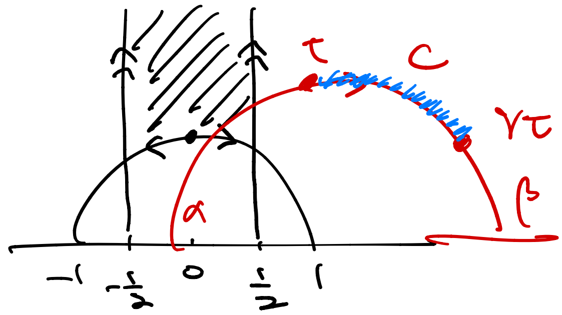
joint w/ Ryotaro Sakamoto

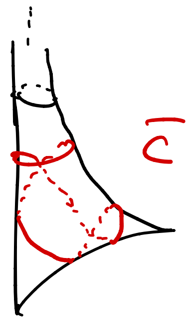
alg. cv./$\mathbb{C}$, $\mathbb{Q}$. &

§ Closed geodesics on the modular curve

$\mathbb{H} := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \} \to SL_2(\mathbb{Z}) \backslash \mathbb{H}$: modular curve

$\circlearrowright SL_2(\mathbb{Z})$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$





$\rightsquigarrow \bar{c}$: closed $\iff \alpha \in \mathbb{Q}(\sqrt{d})$ $\exists d \in \mathbb{Z}_{>0}$ not square

$\beta = \alpha'$: conj of α.

$\Updownarrow \qquad \Updownarrow$

$\exists \gamma \in SL_2(\mathbb{Z})$ $|\mathrm{tr}(\gamma)| > 2$

$\gamma\alpha = \alpha$, $\gamma\beta = \beta$

$\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \dfrac{a\alpha+b}{c\alpha+d} = \alpha \iff c\alpha^2 - (a-d)\alpha - b = 0. \right)$

$\longleftrightarrow$ $\mathbb{Q}(\sqrt{d})$ の数論.

§ Homology classes

regarded as an orbifold

$[\bar{c}] \in H_1(SL_2(\mathbb{Z}) \backslash \mathbb{H}, \mathbb{Z}) = \mathrm{fin}$

$\rightsquigarrow [\bar{c} \otimes (\mathrm{poly})] \in H_1(SL_2(\mathbb{Z}) \backslash \mathbb{H}, \mathrm{Sym}^{2k-2} \mathbb{Z}^2)$

$$\mathrm{Sym}^{2k-2}\mathbb{Z}^2 \cong \{P(X,Y)\in\mathbb{Z}[X,Y] \mid \text{homog. deg } 2k-2\}$$

$$\circlearrowleft$$

$$SL_2(\mathbb{Z}) \qquad (\gamma P)(X,Y) = P\left(\gamma^{-1}\binom{X}{Y}\right)$$

FIX $\Gamma := SL_2(\mathbb{Z})$.

$$M_{2k-2} := \mathrm{Sym}^{2k-2}\mathbb{Z}^2$$

$$I := \ker\left(\mathbb{Z}[\Gamma] \xrightarrow{\text{aug}} \mathbb{Z} : \textstyle\sum c_\gamma[\gamma] \mapsto \sum c_\gamma\right)$$

$$= \langle [\gamma]-[\mathrm{id}] \mid \gamma\in\Gamma \rangle_{\mathbb{Z}}$$

$$\rightsquigarrow H_1(\Gamma\backslash\mathbb{H}, M_{2k-2}) \cong \ker\left(I \otimes_{\mathbb{Z}[\Gamma]} M_{2k-2} \to M_{2k-2}\right)$$

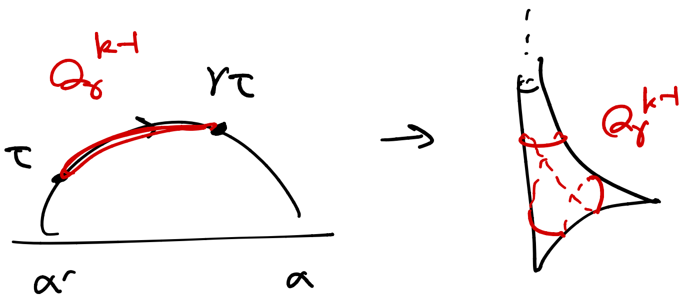
$$([\gamma]-[\mathrm{id}])\otimes P \mapsto \gamma P - P$$

Now, $\gamma\in\Gamma$: hyperbolic i.e. $|\mathrm{tr}(\gamma)| > 2$

$$\rightsquigarrow Q_\gamma(X,Y) := \frac{\mathrm{sgn}(a+d)}{\gcd(c, a-d, b)}\left(cX^2 - (a-d)XY - bY^2\right)$$

$$\rightsquigarrow \gamma\cdot Q_\gamma = Q_\gamma$$

$$\rightsquigarrow z_{k-1}(\gamma) := ([\gamma]-[\mathrm{id}])\otimes Q_\gamma^{k-1} \in H_1(\Gamma\backslash\mathbb{H}, M_{2k-2})$$



$$Z_{2k-2} := \langle z_{k-1}(\gamma) \mid \gamma\in\Gamma : \text{hyp} \rangle_{\mathbb{Z}} \subset H_1(\Gamma\backslash\mathbb{H}, M_{2k-2})$$

Q. How big/small is Z_{2k-2} ?

$\to$ Application to zeta values

§ Zeta values.

• Riemann zeta

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad s \in \mathbb{C} \quad \mathrm{Re}(s) > 1 \rightsquigarrow \text{cont. merom to } s \in \mathbb{C}.$$

s	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\zeta(s)$	$-\frac{1}{252}$	0	$\frac{1}{120}$	0	$-\frac{1}{12}$	$-\frac{1}{2}$	pole	$\frac{\pi^2}{6}$	?	$\frac{\pi^4}{90}$	??	$\frac{\pi^6}{945}$

$\mathbb{Q}$.

Thm (Euler) $k \in \mathbb{Z}_{\geq 1}$.

$$\zeta(2k) \in \mathbb{Q}\pi^{2k} \iff \zeta(1-2k) \in \mathbb{Q}^\times.$$

$$\zeta(2k+1) = ? \iff \zeta(-2k) = 0$$

• 実2次体の zeta. (in terms of $\gamma \in \Gamma$: hyp)

$\gamma \in \Gamma$ hyp $\rightsquigarrow {}^t\gamma^{-1}$ hyp $\rightsquigarrow Q_{{}^t\gamma^{-1}}$ as before.

$$\rightsquigarrow \zeta_\gamma(s) := \frac{1}{2} \sum_{\substack{x \in \mathbb{Z}^2/\langle {}^t\gamma^{-1} \rangle \\ Q_{{}^t\gamma^{-1}}(x) > 0}} \frac{1}{Q_{{}^t\gamma^{-1}}(x)^s} \qquad \mathrm{Re}(s) > 1 \rightsquigarrow s \in \mathbb{C} \text{ merom.}$$

eg. $\gamma = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \to {}^t\gamma^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$

$$\to Q_{{}^t\gamma^{-1}}(X, Y) = X^2 + 3XY + Y^2$$

$$\to \zeta_\gamma(s) = \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2/\langle {}^t\gamma^{-1} \rangle \\ m^2+3mn+n^2 > 0}} \frac{1}{(m^2+3mn+n^2)^s}$$

$$= \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ m \geq 0,\, n > 0}} \frac{1}{(m^2+3mn+n^2)^s}$$

s	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\zeta_\gamma(s)$	$\frac{67}{630}$	0	$\frac{1}{60}$	0	$\frac{1}{30}$	0	pole	$\frac{2\pi^4}{75\sqrt{5}}$	??	$\frac{4\pi^8}{16875\sqrt{5}}$	??	$\frac{536\pi^{12}}{221484375\sqrt{5}}$

Thm (Siegel) $\gamma \in \Gamma$: hyp $k \in \mathbb{Z}_{\geq 1}$

$$\zeta_\gamma(1-k) \in \mathbb{Q}.$$

Q How big are $\mathrm{denom}(\zeta_\gamma(1-k))$?

Conj (Duke 2022) $\gamma \in \Gamma$: hyp $k \in \mathbb{Z}_{\geq 2}$.

$$\mathrm{denom}(\zeta_\gamma(1-k)) \mid \mathrm{denom}(\zeta(1-2k))$$

(divide)

$$\left(\Leftrightarrow \langle \zeta_\gamma(1-k) \mid \gamma \in \Gamma : \mathrm{hyp} \rangle_{\mathbb{Z}} \subset \frac{1}{\mathrm{denom}(\zeta(1-2k))}\mathbb{Z} \subset \mathbb{Q} \right)$$

Main Thm. (B.–Sakamoto)

$$\langle \zeta_\gamma(1-k) \mid \gamma \in \Gamma : \mathrm{hyp} \rangle_{\mathbb{Z}} = \frac{1}{\mathrm{denom}(\zeta(1-2k))}\mathbb{Z}$$

$$\left(\Leftrightarrow \left\langle \frac{\zeta_\gamma(1-k)}{\zeta(1-2k)} \;\middle|\; \gamma \in \Gamma : \mathrm{hyp} \right\rangle_{\mathbb{Z}} = \frac{1}{\mathrm{num}(\zeta(1-2k))}\mathbb{Z} \right)$$

§ Strategy of Pf.

$\boxed{\text{Key:}}$ $\displaystyle\int_{\text{closed geod.}}$ Eisenstein series $= \dfrac{\zeta_\gamma(1-k)}{\zeta(1-2k)}$

$\parallel$

$\langle z_{2k-2}(\gamma), \text{"Eis"} \rangle$ ← study algebraically.

Eisenstein series $k \in \mathbb{Z}_{\geq 2}$.

$$E_{2k}(z) := 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, e^{2\pi i n z} \; : \; \mathbb{H} \to \mathbb{C}$$

$$\left(\sigma_{2k-1}(n) := \sum_{0<d \mid n} d^{2k-1} \right)$$

: Eis series of wt $2k$. (mod. form)

Thm. (Hecke, Siegel ?) $\gamma \in \Gamma$ = hyp $\quad \tau \in \mathbb{H}$.

$$\int_{\tau}^{\gamma\tau} Q_\gamma(z,1)^{k-1} E_{2k}(z)\, dz = (-1)^{k-1} \frac{\zeta_\gamma(1-k)}{\zeta(1-2k)}$$

$$\underbrace{z_{2k-2}(\gamma) = ([\gamma]-[\mathrm{id}]) \otimes Q_\gamma^{k-1}}_{\in\, H_1(\Gamma\backslash\mathbb{H}, M_{2k-2})} \times \underbrace{\mathrm{Eis}_{2k-2}}_{\in\, H^1(\Gamma\backslash\mathbb{H},\, M_{2k-2}^\vee \otimes \mathbb{C})} \to \mathbb{C}$$

(dR)

$\boxed{\mathrm{Eis}}$

Thm (Harder, unpublished)

$$\langle - , \mathrm{Eis}_{2k-2} \rangle : H_1(\Gamma\backslash\mathbb{H}, M_{2k-2}) \twoheadrightarrow \frac{1}{\mathrm{num}(\zeta(1-2k))}\mathbb{Z} \subset \mathbb{C}$$

Now.

$$Z_{2k-2} \subset H_1(\Gamma\backslash\mathbb{H}, M_{2k-2}), \qquad \varphi : Z_{2k-2} \to \frac{1}{\mathrm{num}(\zeta(1-2k))}\mathbb{Z}$$

$$z_{2k-2}(\gamma) \in Z_{2k-2}, \qquad \varphi : z_{2k-2}(\gamma) \mapsto (-1)^{k-1} \frac{\zeta_\gamma(1-k)}{\zeta(1-2k)} \in \frac{1}{\mathrm{num}(\zeta(1-2k))}\mathbb{Z}$$

☆ Duke's conj ✓

☆ Main Thm $\Leftrightarrow$ φ : surj. .... size of Z_{2k-2} matters!

☆ If $Z_{2k-2} = H_1(\Gamma\backslash\mathbb{H}, M_{2k-2}) \Rightarrow$ done.

BUT : It seemed $Z_{2k-2} \neq H_1(\Gamma\backslash\mathbb{H}, M_{2k-2})$ in general.

Solution : prove Z_{2k-2} generated suff large sub-sp of $H_1(\Gamma\backslash\mathbb{H}, M_{2k-2})$

"p-ordinary part" $\forall p$: prime

"good subsp" wrt T_p : Hecke operator

using Hida theory. $(\Gamma_0(p) \subset \Gamma)$

(reduce to "$Z_{2k-2} \bmod p = H_1(\Gamma_0(p)\backslash\mathbb{H}, \mathbb{Z}/p)$")

$\Rightarrow$ surj of φ

$\Rightarrow$ Main Thm $\square$

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§ Next Q. : Beyond ord part ?

$\Downarrow$ numerical experiment.

Observation. $p \geq 5$ prime $k = p+1$ case

$$H_1(\Gamma\backslash\mathbb{H}, M_{2p}) / Z_{2p} \otimes \mathbb{Z}_p \overset{?}{\cong} (\mathbb{Z}/p)^{\oplus 2\dim S_{2p+2}}$$ 22

S_{2p+2} : sp of cusp mod. forms of wt $2p+2$

Prop. $p \geq 5$: prime

$$(\mathbb{Z}/p)^{\oplus \dim S_{2p+2}} \subset H_1(\Gamma\backslash\mathbb{H}, M_{2p}) / Z_{2p} \otimes \mathbb{Z}_p$$