ON THE CONICAL ZETA VALUES AND THE DEDEKIND ZETA VALUES FOR TOTALLY REAL FIELDS

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Abstract. The conical zeta values are a generalization of the multiple zeta values which are defined by certain multiple sums over convex cones. In this paper, we present a relation between the values of the Dedekind zeta functions for totally real fields and the conical zeta values for certain algebraic cones. More precisely, we show that the values of the partial zeta functions for totally real fields can be expressed as a rational linear combination of the conical zeta values associated with certain algebraic cones up to the square root of the discriminant.

1. Introduction

1.1. **The conical zeta values.** Let $n \geq 1$ be an integer. For a subset $C \subset \mathbb{R}^n_{>0}$ and a multi-index $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 1}$, set

$$\zeta_C(\boldsymbol{k}) := \sum_{x \in C \cap \mathbb{Z}^n} \frac{1}{x^{\boldsymbol{k}}}$$

whenever the sum is convergent, where $x^{\boldsymbol{k}} := x_1^{k_1} \cdots x_n^{k_n}$ for $x = {}^t(x_1, \ldots, x_n) \in C \cap \mathbb{Z}^n$. In the case where $C \subset \mathbb{R}^n_{>0}$ is a cone (cf. Definition 2.1), the value $\zeta_C(\boldsymbol{k})$ is often called the *conical zeta value* associated with the cone C, cf. [6], [10].

The following are the basic examples of the conical zeta values and are the motivation for the definition.

Example 1.1. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$, i.e., $e_i = {}^t(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$.

(1) If $C = \sum_{i=1}^n \mathbb{R}_{>0} e_i = \mathbb{R}^n_{>0}$, then

$$\zeta_C(\boldsymbol{k}) = \sum_{x \in \mathbb{Z}^n_{>0}} \frac{1}{x^{\boldsymbol{k}}} = \zeta(k_1) \cdots \zeta(k_n)$$

is the product of Riemann zeta functions.

(2) If $C = \sum_{i=1}^n \mathbb{R}_{>0} \sum_{j=i}^n e_j = \mathbb{R}_{>0}(e_1 + \cdots + e_n) + \mathbb{R}_{>0}(e_2 + \cdots + e_n) + \cdots + \mathbb{R}_{>0} e_n$, then

$$\zeta_C(\boldsymbol{k}) = \sum_{x_1 < x_2 < \cdots < x_n} \frac{1}{x^{\boldsymbol{k}}} = \zeta(k_1, \ldots, k_n)$$

is the multiple zeta value.

More generally, for any rational cone $C \subset \mathbb{R}^n_{>0}$, i.e., C is of the form

$$C = \sum_{i=1}^r \mathbb{R}_{>0} \alpha_i$$

with some rational vectors $\alpha_1, \ldots, \alpha_r \in \mathbb{Q}^n$, the following is known about the values $\zeta_C(\boldsymbol{k})$.

Theorem 1.2 (Terasoma [10])**.** *For a rational cone $C \subset \mathbb{R}^n_{>0}$, the conical zeta value $\zeta_C(\boldsymbol{k})$ can be written as a $\mathbb{Q}^{ab}$-linear combination of the cyclotomic multiple zeta values.*

On the other hand, to the best of the author's knowledge, it seems that little is known about the arithmetic properties of the conical zeta values associated with non-rational cones. In this paper, we consider the conical zeta values for certain algebraic cones which are not necessarily rational, and show that certain $\mathbb{Q}$-linear combinations of such conical zeta values describe the values of the partial zeta functions of totally real fields (up to the square root of the discriminant). Here, we say that a cone $C \subset \mathbb{R}^n_{>0}$ is algebraic if it is of the form

$$C = \sum_{i=1}^{r} \mathbb{R}_{>0}\alpha_i$$

with some algebraic vectors $\alpha_1, \dots, \alpha_r \in (\mathbb{R} \cap \overline{\mathbb{Q}})^n$.

1.2. **Main result.** More precisely, the following is the main theorem of this paper.

Theorem 1.3 (cf. Theorem 2.3)**.** *Let F be a totally real number field of degree $n \geq 1$, and let $\mathfrak{a} \subset F$ be a fractional ideal of F. Then there exist a finite number of algebraic cones $C_1, \dots, C_m \subset \mathbb{R}^n_{>0}$ such that for any $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\zeta_{F,+}(\mathfrak{a}^{-1}, k) \in \frac{1}{\sqrt{d_F}} \sum_{i=1}^{m} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^n_{\geq 1} \\ |\boldsymbol{k}| = nk}} \mathbb{Q}\zeta_{C_i}(\boldsymbol{k}),$$

where $\zeta_{F,+}(\mathfrak{a}^{-1}, s)$ is the (narrow) partial zeta function associated with $\mathfrak{a}^{-1}$, d_F is the discriminant of F, and $|\boldsymbol{k}| := k_1 + \cdots + k_n$ for $\boldsymbol{k} = (k_1, \dots, k_n)$.

Actually, the algebraic cones $C_1, \dots, C_m \subset \mathbb{R}^n_{>0}$ and the coefficients of each $\zeta_{C_i}(\boldsymbol{k})$ can be computed by using the so-called Shintani cone decomposition.

Example 1.4 (cf. Section 3.1)**.** Let $F = \mathbb{Q}(\sqrt{5})$, and let $\mathfrak{a} = \mathcal{O}_F = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Moreover, let

$$\begin{aligned} C &= \mathbb{R}_{>0}\begin{pmatrix} \frac{3+\sqrt{5}}{2} \\ 1 \end{pmatrix} + \mathbb{R}_{>0}\begin{pmatrix} \frac{3-\sqrt{5}}{2} \\ 1 \end{pmatrix} \\ &= \{{}^t(x_1, x_2) \in \mathbb{R}^2_{>0} \mid -x_1^2 + 3x_1x_2 - x_2^2 > 0\} \end{aligned}$$

be an algebraic cone generated by ${}^t\left(\frac{3+\sqrt{5}}{2}, 1\right)$ and ${}^t\left(\frac{3-\sqrt{5}}{2}, 1\right)$. Then we can prove that

$$\zeta_{\mathbb{Q}(\sqrt{5})}(k) = \zeta_{F,+}(\mathfrak{a}^{-1}, k) \in \frac{1}{\sqrt{5}} \sum_{\substack{(k_1,k_2) \in \mathbb{Z}^2_{\geq 1} \\ k_1+k_2=2k}} \mathbb{Q}\zeta_C(k_1, k_2) \tag{1.1}$$

for all $k \in \mathbb{Z}_{\geq 2}$, where $\zeta_{\mathbb{Q}(\sqrt{5})}(s)$ is the Dedekind zeta function for $\mathbb{Q}(\sqrt{5})$. For example, we find

$$\zeta_{\mathbb{Q}(\sqrt{5})}(2) = \frac{1}{5\sqrt{5}}(4\zeta_C(3,1) + 3\zeta_C(2,2)), \tag{1.2}$$

$$\zeta_{\mathbb{Q}(\sqrt{5})}(3) = \frac{1}{25\sqrt{5}}(12\zeta_C(5,1) + 18\zeta_C(4,2) + 11\zeta_C(3,3)). \tag{1.3}$$

Remark 1.5.
(1) Note that the theorem of Siegel-Klingen describes all the critical values of the partial zeta functions in a more beautiful way, e.g., $\zeta_{\mathbb{Q}(\sqrt{5})}(2) = \frac{2\pi^4}{75\sqrt{5}}$. On the other hand, one feature of the above theorem is that it describes both critical and non-critical values in a uniform way using the conical zeta values associated with algebraic cones.
(2) Recently, I found that Duke has also obtained the same formula in the case of real quadratic fields in the study of higher Rademacher symbols [5].

In the next section (Section 2), after fixing some notation and convention, we state our main theorem (in a slightly more precise way), and then prove the main theorem. The proof is really very simple and elementary. We start from an integral representation of the values of the partial zeta functions which can be seen as a variant of the classical Hecke integral formula, that is, a formula which expresses the values of the partial zeta functions of real quadratic fields as an integral of the Eisenstein series along a closed geodesics on the modular curve. One key point of the integral representation in this paper is that we consider a "partial" Eisenstein series which can be seen as a decomposed piece of the Eisenstein series along cones (cf. Remark 2.8 (1)). Then by using some other also classical techniques such as the unfolding of the integrals and the Shintani cone decomposition, we prove our theorem. In Section 3, we present some examples to illustrate our main theorem.

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2. Main Theorem

Conventions.

- We fix an integer $n \geq 1$ throughout the paper.
- For a matrix A, its transpose is denote by ${}^tA$.
- For a ring R, elements in R^r are basically regarded as column vectors, and the matrix algebra $M_r(R)$ acts on R^r by the matrix multiplication form the left.
- For vectors $v_1, \dots, v_{r'} \in R^r$, we often regard the r'-tuple $I = (v_1, \dots, v_{r'}) \in (R^r)^{r'}$ as an $r \times r'$-matrix whose columns are $v_1, \dots, v_{r'}$.
- For vectors $x = {}^t(x_1, \dots, x_n), y = {}^t(y_1, \dots, y_n)$, the bracket $\langle x, y \rangle := x_1y_1 + \cdots + x_ny_n$ denotes the dot product of x and y.

2.1. **Cones and conical zeta values.** First we fix some notation and terminologies concerning cones that will be used in this paper.

For $r \geq 0$, $I = (\alpha_1, \dots, \alpha_r) \in (\mathbb{R}^n - \{0\})^r$, we set

$$C_I := \sum_{i=1}^{r} \mathbb{R}_{>0}\alpha_i \subset \mathbb{R}^n. \tag{2.1}$$

In the case where $r = 0, I = \emptyset$, we set $C_\emptyset := \{0\}$.

Definition 2.1. (1) A subset $C \subset \mathbb{R}^n$ is called an *open convex polyhedral cone* if C is of the form $C = C_I$ for some $r \geq 0$, $I = (\alpha_1, \dots, \alpha_r) \in (\mathbb{R}^n - \{0\})^r$. In this case, we say that C is generated by $\alpha_1, \dots, \alpha_r$. Since in this paper we basically deal only with open convex polyhedral cones, by abuse of notation, we will refer to an open convex polyhedral cone simply as a *cone*.

(2) A cone $C \subset \mathbb{R}^n$ is said to be *simplicial* if we can take linearly independent generators of C, i.e., there exists $I = (\alpha_1, \dots, \alpha_r) \in (\mathbb{R}^n - \{0\})^r$ such that $C = C_I$ and $\alpha_1, \dots, \alpha_r$ are linearly independent over $\mathbb{R}$.

(3) A cone $C \subset \mathbb{R}^n$ is said to be *rational* if we can take rational generators of C, i.e., there exists $I = (\alpha_1, \dots, \alpha_r) \in (\mathbb{Q}^n - \{0\})^r$ such that $C = C_I$.

(4) More generally, let $K \subset \mathbb{R}$ be a subfield. Then a cone $C \subset \mathbb{R}^n$ is said to be *K-rational* if there exists $I = (\alpha_1, \dots, \alpha_r) \in (K^n - \{0\})^r$ such that $C = C_I$.

(5) A cone $C \subset \mathbb{R}^n$ is said to be *algebraic* if there exists an algebraic subfield $K \subset \mathbb{R} \cap \overline{\mathbb{Q}}$ such that C is K-rational.

(6) A cone $C \subset \mathbb{R}^n$ is said to be *smooth* if there exists $I = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{Z}^n - \{0\})^r$ which can be extended to a basis of $\mathbb{Z}^n$ such that $C = C_I$. In particular, a smooth cone is a rational simplicial cone.
(7) A cone $C \subset \mathbb{R}^n$ is said to be *totally positive* if $C \subset \mathbb{R}^n_{>0}$.

Definition 2.2. Let $C \subset \mathbb{R}^n_{>0}$ be a totally positive (open convex polyhedral) cone, and let $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 1}$ be a multi-index. Then we define the conical zeta value associated with the cone C with index $\boldsymbol{k}$ to be

$$\zeta_C(\boldsymbol{k}) := \sum_{x \in C \cap \mathbb{Z}^n} \frac{1}{x^{\boldsymbol{k}}}$$

whenever the sum is convergent, where $x^{\boldsymbol{k}} := x_1^{k_1} \cdots x_n^{k_n}$ for $x = {}^t(x_1, \ldots, x_n) \in C \cap \mathbb{Z}^n$.

2.2. **Statement of the main theorem.** Let F be a totally real field of degree n, and let

$$\tau_1, \ldots, \tau_n \colon F \hookrightarrow \mathbb{R}$$

be the field embeddings of F into $\mathbb{R}$. Moreover, we put $F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R}$. Then $\tau_1, \ldots, \tau_n$ induces an isomorphism

$$\tau = (\tau_1, \ldots, \tau_n) \colon F_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n.$$

For a subset $A \subset F_{\mathbb{R}}$, we denote by A_+ its totally positive part, i.e.,

$$A_+ := \{a \in A \mid \forall i \in \{1, \ldots, n\}, \tau_i(a) > 0\}.$$

Let $\mathcal{O}_F$ denotes the ring of integers of F, and let $\mathfrak{a} \subset F$ be a fractional ideal of F. Then we define the (narrow) partial zeta function associated with $\mathfrak{a}^{-1}$ to be

$$\zeta_{F,+}(\mathfrak{a}^{-1}, s) := N\mathfrak{a}^s \sum_{x \in \mathfrak{a}_+/\mathcal{O}_{F,+}^\times} \frac{1}{N_{F/\mathbb{Q}}(x)^s}$$

for $\mathrm{Re}(s) > 1$, where $N\mathfrak{a}$ is the norm of the fractional ideal $\mathfrak{a}$ and $N_{F/\mathbb{Q}}$ is the norm of the field extension $F/\mathbb{Q}$.

Finally, we define

$$F' := \tau_1(F) \cdots \tau_n(F) \subset \mathbb{R}$$

to be the subfield of $\mathbb{R}$ generated by $\tau_1(F), \ldots, \tau_n(F)$.

The following is the main theorem of this paper.

Theorem 2.3. *Let F and $\mathfrak{a}$ be as above. Then there exist a finite number of totally positive F'-rational cones $C_1, \ldots, C_m \subset \mathbb{R}^n_{>0}$ such that for any $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\zeta_{F,+}(\mathfrak{a}^{-1}, k) \in \frac{1}{\sqrt{d_F}} \sum_{i=1}^{m} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^n_{\geq 1} \\ |\boldsymbol{k}| = nk}} \mathbb{Q}\zeta_{C_i}(\boldsymbol{k}), \tag{2.2}$$

where d_F is the discriminant of F and $|\boldsymbol{k}| := k_1 + \cdots + k_n$ for $\boldsymbol{k} = (k_1, \ldots, k_n)$ is the weight of the multi-index. In other words, the value $\zeta_{F,+}(\mathfrak{a}^{-1}, k)$ of the partial zeta function at k can be written as a rational linear combination of the conical zeta values associated with F'-rational cones with indices of weight nk divided by the square root of the discriminant.

Remark 2.4. The F'-rational cones $C_1, \ldots, C_m \subset \mathbb{R}^n_{>0}$ and the coefficients of $\zeta_{C_i}(\boldsymbol{k})$ in (2.2) can be computed using the Shintani cone decomposition (cf. (2.10) and Section 3), but they are not necessarily unique.

Remark 2.5. This theorem shows the connection between the conical zeta values associated with algebraic cones and the values of the partial zeta functions of totally real fields. On the other hand, it does not tell us about the properties of individual conical zeta values. It might be an interesting problem to consider the arithmetic properties of the individual conical zeta values associated with algebraic cones.

The proof of this theorem will be given in Section 2.4. To this end, we first prepare an integral representation of the values of the partial zeta functions (Proposition 2.11). We will use similar arguments and notation to [2, Section 2, Section 7].

2.3. **The Hecke type integral representation.** Let us take a basis $w_1, \dots, w_n \in \mathfrak{a}$ of $\mathfrak{a}$ over $\mathbb{Z}$, and set

$$w := {}^t(w_1, \dots, w_n) \in \mathfrak{a}^n \subset F^n,$$
$$w^{(i)} := \tau_i(w) = {}^t(\tau_i(w_1), \dots, \tau_i(w_n)) \in \mathbb{R}^n$$

for $i = 1, \dots, n$, and

$$N_w(x) := \prod_{i=1}^{n} \langle x, w^{(i)} \rangle \in \mathbb{Q}[x_1, \dots, x_n]$$

for $x = {}^t(x_1, \dots, x_n)$. By replacing w_1 with $-w_1$ if necessary, we assume $\det(w^{(1)}, \dots, w^{(n)}) > 0$, where $(w^{(1)}, \dots, w^{(n)})$ is regarded as an $n \times n$-matrix. Then we have

$$\det(w^{(1)}, \dots, w^{(n)}) = \sqrt{d_F} N\mathfrak{a},$$

where d_F is the discriminant of F and $N\mathfrak{a}$ is the norm of the fractional ideal $\mathfrak{a}$. Note also that w defines an isomorphism

$$\begin{array}{ccc} \langle -, w \rangle \colon \mathbb{Q}^n & \xrightarrow{\sim} & F; \ x \mapsto \langle x, w \rangle \\ \cup & & \cup \\ \mathbb{Z}^n & \xrightarrow{\sim} & \mathfrak{a} \end{array} \tag{2.3}$$

which extends also to $\langle -, w \rangle \colon \mathbb{R}^n \xrightarrow{\sim} F_{\mathbb{R}}$, and N_w is the norm map with respect to this isomorphism, i.e., we have $N_w(x) = N_{F/\mathbb{Q}}(\langle x, w \rangle)$ for $x \in \mathbb{R}^n$. We define

$$\begin{aligned} T_{w,+} :=& \{x \in \mathbb{R}^n \mid \langle x, w \rangle \in F_{\mathbb{R},+}\} \\ =& \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, w^{(i)} \rangle > 0\} \end{aligned}$$

to be the subset of $\mathbb{R}^n$ corresponding to the set $F_{\mathbb{R},+}$ of totally positive elements in $F_{\mathbb{R}}$ under this isomorphism.

Moreover, let

$$\rho_w \colon F^\times \to GL_n(\mathbb{Q})$$

be the regular representation of $F^\times$ on F with respect to the isomorphism (2.3), i.e.,

$$\langle \rho_w(\alpha) x, w \rangle = \alpha \langle x, w \rangle \in F$$

for $\alpha \in F^\times$ and $x \in \mathbb{Q}^n$, and set

$$\Gamma_{w,+} := \rho_w(\mathcal{O}_{F,+}^\times) \subset SL_n(\mathbb{Z})$$

to be the subgroup of $SL_n(\mathbb{Z})$ corresponding to the totally positive units of $\mathcal{O}_F$ under the map ρ_w. Note that $\Gamma_{w,+}$ acts on $T_{w,+} \cap \mathbb{Z}^n$.

Now, let $w_1^*, \dots, w_n^* \in F$ be the dual basis of $w_1, \dots, w_n$ with respect to the trace $Tr_{F/\mathbb{Q}}$ of the field extension $F/\mathbb{Q}$, i.e.,

$$Tr_{F/\mathbb{Q}}(w_i w_j^*) = \delta_{ij}$$

for $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta. Then it is known that $w_1^*, \dots, w_n^*$ form a basis of the fractional ideal

$$\mathfrak{a}^* = \{\alpha \in F \mid Tr_{F/\mathbb{Q}}(\alpha\mathfrak{a}) \subset \mathbb{Z}\} \subset F.$$

Repeating the above construction, we define

$$\begin{aligned} w^* &:= {}^t(w_1^*, \dots, w_n^*) \in F^n, \\ w^{*(i)} &:= \tau_i(w^*) = {}^t(\tau_i(w_1^*), \dots, \tau_i(w_n^*)) \in \mathbb{R}^n, \\ N_{w^*}(x) &:= \prod_{i=1}^n \langle x, w^{*(i)} \rangle \in \mathbb{Q}[x_1, \dots, x_n], \\ \rho_{w^*} &: F^\times \to GL_n(\mathbb{Q}) \end{aligned}$$

starting from the dual basis $w_1^*, \dots, w_n^*$.

Lemma 2.6. (1) *$w^{*(1)}, \dots, w^{*(n)}$ are the dual basis of $w^{(1)}, \dots, w^{(n)}$ with respect to $\langle -, - \rangle$, i.e., we have*

$$\langle w^{(i)}, w^{*(j)} \rangle = \delta_{ij}.$$

(2) *We have*

$$\begin{aligned} C_{(w^{(1)}, \dots, w^{(n)})} &= \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, w^{*(i)} \rangle > 0\}, \\ C_{(w^{*(1)}, \dots, w^{*(n)})} &= \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, w^{(i)} \rangle > 0\} = T_{w,+}. \end{aligned}$$

Here recall that $C_{(w^{(1)}, \dots, w^{(n)})}$ (resp. $C_{(w^{*(1)}, \dots, w^{*(n)})}$) denotes the cone generated by the vectors $w^{(1)}, \dots, w^{(n)}$ (resp. $w^{*(1)}, \dots, w^{*(n)}$), cf. (2.1). *In particular, we see that $T_{w,+}$ is an F'-rational cone.*

(3) *For $\alpha \in F^\times$ and $i \in \{1, \dots, n\}$, we have*

$$\rho_w(\alpha) w^{*(i)} = \tau_i(\alpha) w^{*(i)},$$

i.e., $w^{*(i)}$ is an eigenvector of $\rho_w(\alpha)$ with eigenvalue $\tau_i(\alpha)$.

(4) *For $\gamma \in \Gamma_{w,+}$, we have*

$$N_{w^*}({}^t\gamma x) = N_{w^*}(x).$$

Proof. (1) Set

$$\begin{aligned} W &:= (w^{(1)}, \dots, w^{(n)}) \in GL_n(\mathbb{R}), \\ W^* &:= (w^{*(1)}, \dots, w^{*(n)}) \in GL_n(\mathbb{R}). \end{aligned}$$

Then since $w_1^*, \dots, w_n^*$ are the dual basis of $w_1, \dots, w_n$ with respect to $Tr_{F/\mathbb{Q}}$, we find

$$W^* {}^tW = (Tr_{F/\mathbb{Q}}(w_i^* w_j))_{ij} = (\delta_{ij})_{ij},$$

and hence

$$(\langle w^{(i)}, w^{*(j)} \rangle)_{ij} = {}^tW W^* = (\delta_{ij})_{ij}.$$

The assertions (2) and (3) follow from (1), and (4) follows from (3). □

Next, we consider the following infinite series.

Definition 2.7. For $k \in \mathbb{Z}_{\geq 1}$, a subset $C \subset \mathbb{R}^n - \{0\}$, and $y \in \mathbb{C}^n - \{0\}$, set

$$\psi_{nk,C}(y) := \sum_{x \in C \cap \mathbb{Z}^n} \frac{1}{\langle x, y \rangle^{n+nk}}$$

whenever the sum is convergent.

Remark 2.8.
(1) In the case where $n = 2$, $y = {}^t(z, 1)$ with $z \in \mathfrak{H}$ (the upper half plane), the series

$$\psi_{2k,C}(y) = \sum_{{}^t(x_1,x_2) \in C \cap \mathbb{Z}^2} \frac{1}{(x_1 z + x_2)^{2+2k}}$$

can be seen as a "fragment" of the holomorphic Eisenstein series of weight $2 + 2k$, in which the sum is restricted to the subset C.

(2) In [2], we have considered the case where C is (the Q-perturbation of) a rational cone, in which case $\psi_{nk,C}$ is essentially a finite sum of the Barnes zeta functions. In this paper, we will consider the case where C is an algebraic cone $T_{w,+}$.

Proposition 2.9. *For $k \in \mathbb{Z}_{\geq 1}$, the infinite series*

$$\psi_{nk,T_{w,+}}(y) = \sum_{x \in T_{w,+} \cap \mathbb{Z}^n} \frac{1}{\langle x, y \rangle^{n+nk}}.$$

converges absolutely and locally uniformly on

$$\widetilde{Y}_{w,+} := \{y \in \mathbb{C}^n - \{0\} \mid \exists \lambda \in \mathbb{C}^\times, \forall i \in \{1, \ldots, n\}, \mathrm{Re}(\langle w^{*(i)}, \lambda y \rangle) > 0\}.$$

Proof. This can be proved in a straightforward way. Here we give a proof which uses [2, Proposition 6.1.2]. First, note that it suffices to prove that $\psi_{nk,T_{w,+}}(y)$ converges absolutely and locally uniformly on

$$V := \{y \in \mathbb{C}^n - \{0\} \mid \forall i \in \{1, \ldots, n\}, \mathrm{Re}(\langle w^{*(i)}, y \rangle) > 0\}.$$

Let $y' = {}^t(y'_1, \ldots, y'_n) \in V$. By Lemma 2.6 (2), this is equivalent to

$$\mathrm{Re}(y') = {}^t(\mathrm{Re}(y'_1), \ldots, \mathrm{Re}(y'_n)) \in C_{(w^{(1)}, \ldots, w^{(n)})} = \sum_{i=1}^n \mathbb{R}_{>0} w^{(i)}.$$

Then we can take linearly independent $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}^n \cap C_{(w^{(1)}, \ldots, w^{(n)})}$ such that $\mathrm{Re}(y') \in C_{(\alpha, \ldots, \alpha_n)}$. Let $\alpha'_1, \ldots, \alpha'_n \in \mathbb{Q}^n$ be the dual basis of $\alpha_1, \ldots, \alpha_n$ with respect to the dot product $\langle -, - \rangle$. Set $I := (\alpha'_1, \ldots, \alpha'_n)$, and

$$\begin{aligned} V_I :=& \{y \in \mathbb{C}^n - \{0\} \mid \forall i \in \{1, \ldots, n\}, \mathrm{Re}(\langle \alpha'_i, y \rangle) > 0\} \\ =& \{y \in \mathbb{C}^n - \{0\} \mid \mathrm{Re}(y) \in C_{(\alpha, \ldots, \alpha_n)}\}. \end{aligned}$$

Then again by using Lemma 2.6 (2), we see that $y' \in V_I \subset V$ and that $T_{w,+} \subset C_I = \sum_{i=1}^n \mathbb{R}_{>0} \alpha'_i$. Therefore, now it suffices to prove that

$$\psi_{nk,C_I}(y) = \sum_{x \in C_I \cap \mathbb{Z}^n} \frac{1}{\langle x, y \rangle^{n+nk}}$$

converges absolutely and locally uniformly on V_I, and this follows from [2, Proposition 6.1.2]. $\square$

Let ω denote the $(n-1)$-form on $\mathbb{C}^n - \{0\}$ defined by

$$\omega(y) := \sum_{i=1}^n (-1)^{i-1} y_i dy_1 \wedge \cdots \wedge \check{dy_i} \wedge \cdots \wedge dy_n,$$

where $\check{dy_i}$ means that dy_i is omitted. Moreover, let

$$\pi_{\mathbb{C}} : \mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}(\mathbb{C}) := (\mathbb{C}^n - \{0\})/\mathbb{C}^\times$$

denote the natural projection, and set

$$Y_{w,+} := \pi_{\mathbb{C}}(\widetilde{Y}_{w,+}) \subset \mathbb{P}^{n-1}(\mathbb{C}).$$

Note that by Lemma 2.6 (2), (3), we see that $\Gamma_{w,+} \subset SL_n(\mathbb{Z})$ acts on $\widetilde{Y}_{w,+}$ and $Y_{w,+}$ by

$$ {}^t\gamma^{-1} : \widetilde{Y}_{w,+} \xrightarrow{\sim} \widetilde{Y}_{w,+};\ y \mapsto {}^t\gamma^{-1} y \tag{2.4}$$

where ${}^t\gamma^{-1}y$ on the right hand side is the usual matrix action of the transposed inverse of γ from the left.

Corollary 2.10. *For $k \in \mathbb{Z}_{\geq 1}$, the $(n-1)$-form*

$$N_{w^*}(y)^k \psi_{nk,T_{w,+}}(y)\omega(y)$$

on $\widetilde{Y}_{w,+}$ defines a $\Gamma_{w,+}$-invariant closed $(n-1)$-form on $Y_{w,+} \subset \mathbb{P}^{n-1}(\mathbb{C})$. Here the $\Gamma_{w,+}$-invariance means that we have

$$N_{w^*}({}^t\gamma^{-1}y)^k \psi_{nk,T_{w,+}}({}^t\gamma^{-1}y)\omega({}^t\gamma^{-1}y) = N_{w^*}(y)^k \psi_{nk,T_{w,+}}(y)\omega(y)$$

for all $\gamma \in \Gamma_{w,+}$.

Proof. By Proposition 2.9, $N_{w^*}(y)^k \psi_{nk,T_{w,+}}(y)$ defines a holomorphic function on $\widetilde{Y}_{w,+}$ which is homogeneous of degree $-n$, i.e.,

$$N_{w^*}(\lambda y)^k \psi_{nk,T_{w,+}}(\lambda y) = \lambda^{-n} N_{w^*}(y)^k \psi_{nk,T_{w,+}}(y)$$

for $\lambda \in \mathbb{C}^\times$. Therefore, we see that $N_{w^*}(y)^k \psi_{nk,T_{w,+}}(y)\omega(y)$ defines a holomorphic closed $(n-1)$-form on $Y_{w,+} \subset \mathbb{P}^{n-1}(\mathbb{C})$.

To see the $\Gamma_{w,+}$-invariance, let $\gamma \in \Gamma_{w,+}$. First by Lemma 2.6 (4), we have $N_{w^*}({}^t\gamma^{-1}y)^k = N_{w^*}(y)^k$. Moreover, since we see that $\omega(gy) = \det(g)\omega(y)$ for $g \in GL_n(\mathbb{C})$, we have $\omega({}^t\gamma^{-1}y) = \omega(y)$. Finally, since $\Gamma_{w,+}$ is acting on $T_{w,+} \cap \mathbb{Z}^n$, we find

$$\begin{aligned}
\psi_{nk,T_{w,+}}({}^t\gamma^{-1}y) &= \sum_{x \in T_{w,+} \cap \mathbb{Z}^n} \frac{1}{\langle x, {}^t\gamma^{-1}y\rangle^{n+nk}} \\
&= \sum_{x \in T_{w,+} \cap \mathbb{Z}^n} \frac{1}{\langle \gamma^{-1}x, y\rangle^{n+nk}} \\
&= \sum_{x \in T_{w,+} \cap \mathbb{Z}^n} \frac{1}{\langle x, y\rangle^{n+nk}} = \psi_{nk,T_{w,+}}(y).
\end{aligned}$$

This shows the corollary. □

Now, set

$$\Delta^\circ_{n-1} := \left\{ t = {}^t(t_1, \ldots, t_n) \in \mathbb{R}^n_{>0} \;\middle|\; \sum_{i=1}^n t_i = 1 \right\}.$$

We embed Δ°_{n-1} into $\mathbb{R}^{n-1}$ by

$$\Delta^\circ_{n-1} \hookrightarrow \mathbb{R}^{n-1};\ {}^t(t_1, \ldots, t_n) \mapsto {}^t(t_2, \ldots, t_n)$$

and equip Δ°_{n-1} with an orientation induced from the standard orientation on $\mathbb{R}^{n-1}$.

For $I=(\alpha_1,\ldots,\alpha_n)\in(\mathbb{C}^n-\{0\})^n$ such that $\alpha_1,\ldots,\alpha_n$ are a basis of $\mathbb{C}^n$ over $\mathbb{C}$, we define

$$\Delta_I^\circ := \pi_{\mathbb{C}}\left(C_{(\alpha_1,\ldots,\alpha_n)}\right)\subset\mathbb{P}^{n-1}(\mathbb{C})$$

to be the image of the cone $C_{(\alpha_1,\ldots,\alpha_n)}=\sum_{i=1}^n\mathbb{R}_{>0}\alpha_i\subset\mathbb{C}^n-\{0\}$ in $\mathbb{P}^{n-1}(\mathbb{C})$. We have an isomorphism

$$\sigma_I:\Delta_{n-1}^\circ\xrightarrow{\sim}\Delta_I^\circ\subset\mathbb{P}^{n-1}(\mathbb{C});\ t={}^t(t_1,\ldots,t_n)\mapsto\pi_{\mathbb{C}}\left(\sum_{i=1}^n t_i\alpha_i\right),$$

and we equip Δ_I° with an orientation induced from Δ_{n-1}° via this isomorphism.

Let us consider the case where $\alpha_i=w^{(i)}$ and set

$$\Delta_{w,+}^\circ:=\Delta_{(w^{(1)},\ldots,w^{(n)})}^\circ=\pi_{\mathbb{C}}(C_{(w^{(1)},\ldots,w^{(n)})})\subset\mathbb{P}^{n-1}(\mathbb{C}).$$

By Lemma 2.6 (2), we have $C_{(w^{(1)},\ldots,w^{(n)})}\subset\widetilde{Y}_{w,+}$, and hence $\Delta_{w,+}^\circ\subset Y_{w,+}$. Moreover, by Lemma 2.6 (3), we see that the action (2.4) of $\Gamma_{w,+}$ on $\widetilde{Y}_{w,+}$ (resp. $Y_{w,+}$) preserves $C_{(w^{(1)},\ldots,w^{(n)})}$ (resp. $\Delta_{w,+}^\circ$ and its orientation), and hence by Dirichlet's unit theorem we see that $\Gamma_{w,+}\backslash\Delta_{w,+}^\circ$ is a compact oriented manifold of dimension $n-1$.

Then we have the following integral representation of $\zeta_{F,+}(\mathfrak{a}^{-1},k)$.

Proposition 2.11. *For $k\in\mathbb{Z}_{\geq 2}$, we have*

$$\int_{\Gamma_{w,+}\backslash\Delta_{w,+}^\circ}N_{w^*}(y)^{k-1}\psi_{n(k-1),T_{w,+}}(y)\omega(y)=\frac{((k-1)!)^n\sqrt{d_F}}{(nk-1)!N\mathfrak{a}^{k-1}}\zeta_{F,+}(\mathfrak{a}^{-1},k).\tag{2.5}$$

In order to prove this proposition, we recall a classical formula known as the Feynman parametrization.

Proposition 2.12 (Feynman parametrization)**.** *Let $x\in\mathbb{C}^n-\{0\}$, and let $\alpha_1,\ldots,\alpha_n\in\mathbb{C}^n$ be a basis over $\mathbb{C}$ such that $\mathrm{Re}(\langle x,\alpha_i\rangle)>0$ for all $i=1,\ldots,n$. Moreover, let $\alpha_1',\ldots,\alpha_n'\in\mathbb{C}^n$ be the dual basis of $\alpha_1,\ldots,\alpha_n$ with respect to $\langle -,-\rangle$. Set*

$$\Delta_{(\alpha_1,\ldots,\alpha_n)}^\circ:=\pi_{\mathbb{C}}(C_{(\alpha_1,\ldots,\alpha_n)})\subset\mathbb{P}^{n-1}(\mathbb{C}),$$

and equip $\Delta_{(\alpha_1,\ldots,\alpha_n)}^\circ$ with an orientation as above. Then for $\boldsymbol{k}=(k_1,\ldots,k_n)\in\mathbb{Z}_{\geq 0}^n$, we have

$$\int_{\Delta_{(\alpha_1,\ldots,\alpha_n)}^\circ}\langle\alpha_1',y\rangle^{k_1}\cdots\langle\alpha_n',y\rangle^{k_n}\frac{\omega(y)}{\langle x,y\rangle^{n+|\boldsymbol{k}|}}=\frac{\boldsymbol{k}!}{(n+|\boldsymbol{k}|-1)!}\frac{\det(\alpha_1,\ldots,\alpha_n)}{\langle x,\alpha_1\rangle^{k_1+1}\cdots\langle x,\alpha_n\rangle^{k_n+1}},$$

where $|\boldsymbol{k}|=k_1+\cdots+k_n$ and $\boldsymbol{k}!=k_1!\cdots k_n!$.

Proof. See, for example, [7] or [2, Proposition 7.1.3]. □

Proof of Proposition 2.11. Let $A\subset T_{w,+}\cap\mathbb{Z}^n$ be a system of representatives of $\Gamma_{w,+}\backslash(T_{w,+}\cap\mathbb{Z}^n)$. Then by using Proposition 2.12 as well as the $\Gamma_{w,+}$-invariance of N_{w*} (Lemma 2.6 (4)) and ω, we find

$$
\begin{aligned}
&\int_{\Gamma_{w,+}\backslash\Delta^\circ_{w,+}} N_{w^*}(y)^{k-1}\psi_{n(k-1),T_{w,+}}(y)\omega(y)\\
=&\int_{\Gamma_{w,+}\backslash\Delta^\circ_{w,+}} N_{w^*}(y)^{k-1}\sum_{\gamma\in\Gamma_{w,+}}\sum_{x\in A}\frac{1}{\langle\gamma x,y\rangle^{nk}}\omega(y)\\
=&\sum_{x\in A}\int_{\Gamma_{w,+}\backslash\Delta^\circ_{w,+}}\sum_{\gamma\in\Gamma_{w,+}} N_{w^*}({}^t\gamma y)^{k-1}\frac{1}{\langle x,{}^t\gamma y\rangle^{nk}}\omega({}^t\gamma y)\\
=&\sum_{x\in A}\int_{\Delta^\circ_{w,+}} N_{w^*}(y)^{k-1}\frac{1}{\langle x,y\rangle^{nk}}\omega(y)\\
=&\frac{((k-1)!)^n\det(w^{(1)},\dots,w^{(n)})}{(nk-1)!}\sum_{x\in A}\frac{1}{N_w(x)^k}\\
=&\frac{((k-1)!)^n\sqrt{d_F}}{(nk-1)!N\mathfrak{a}^{k-1}}\zeta_{F,+}(\mathfrak{a}^{-1},k),
\end{aligned}
$$

where Proposition 2.12 is used in the fourth equality with $\alpha_i = w^{(i)}$. □

2.4. **Proof of the main theorem.** We also need the following classical fact.

Proposition 2.13 (Shintani cone decomposition)**.** *There exists a finite set*

$$\Phi \subset \coprod_{r=1}^{n}(C_{(w^{(1)},\dots,w^{(n)})}\cap\mathbb{Z}^n)^r$$

satisfying the following conditions:

(i) *For all $I=(\alpha_1,\dots,\alpha_r)\in\Phi$, the vectors $\alpha_1,\dots,\alpha_r\in C_{(w^{(1)},\dots,w^{(n)})}\cap\mathbb{Z}^n$ can be extended to a basis of $\mathbb{Z}^n$ over $\mathbb{Z}$. In particular, the cone C_I is a smooth cone for all $I\in\Phi$.*

(ii) *We have*

$$C_{(w^{(1)},\dots,w^{(n)})} = \coprod_{\gamma\in\Gamma_{w,+}}\coprod_{I\in\Phi}{}^t\gamma C_I.$$

Here $\coprod$ denotes the disjoint union.

Proof. It is well known that there exists Φ with condition (ii), cf. [9]. Then by subdividing each cone if necessary, we achieve the condition (i), cf. [4, Section 11.1]. □

Remark 2.14. Note that there is also a stronger version of this proposition which requires Φ to be a fan, cf. [1, Chapter III, Corollary 7.6], [8]. However, for our purpose, Proposition 2.13 is sufficient.

Proof of Theorem 2.3. We will prove the theorem by computing the left hand side of (2.5) in a different way from Proposition 2.11 using the Shintani cone decomposition.

Take Φ as in Proposition 2.13, and let

$$\Phi^{(n)} := \Phi\cap(C_{(w^{(1)},\dots,w^{(n)})}\cap\mathbb{Z}^n)^n, \tag{2.6}$$

i.e., $\Phi^{(n)}$ is exactly the subset of Φ such that C_I is an n-dimensional cone for $I\in\Phi^{(n)}$. By permuting the order of the vectors if necessary, we may assume

$$\det(I)=\det(\alpha_1,\dots,\alpha_n)=1$$

for all $I=(\alpha_1,\dots,\alpha_n)\in\Phi^{(n)}$. Note that in the case $n=1$, we automatically have $\alpha_1=1$, because we have assumed that $\det(w^{(1)},\dots,w^{(n)})=w^{(1)}>0$ and $\alpha_1\in\mathbb{R}_{>0}w^{(1)}$.

For $I \in \Phi^{(n)}$, note that we have

$$\Delta_I^\circ = \pi_{\mathbb{C}}(C_I) \subset \Delta_{w,+} \subset Y_{w,+},$$

and that the orientation of Δ_I° coincides with the orientation restricted from $\Delta_{w,+}^\circ$ because $\det(I) > 0$ and $\det(w^{(1)}, \ldots, w^{(n)}) > 0$. Therefore, by the conditions (i) and (ii) of Φ, we see that

$$\int_{\Gamma_{w,+}\backslash\Delta_{w,+}^\circ} \eta = \sum_{I\in\Phi^{(n)}} \int_{\Delta_I^\circ} \eta$$

for any $\Gamma_{w,+}$-invariant $(n-1)$-form on $Y_{w,+}$. Hence we find

$$\begin{aligned}&\int_{\Gamma_{w,+}\backslash\Delta_{w,+}^\circ} N_{w^*}(y)^{k-1}\psi_{n(k-1),T_{w,+}}(y)\omega(y)\\ &= \sum_{I\in\Phi^{(n)}} \int_{\Delta_I^\circ} N_{w^*}(y)^{k-1} \sum_{x\in T_{w,+}\cap\mathbb{Z}^n} \frac{1}{\langle x,y\rangle^{nk}}\omega(y).\end{aligned} \tag{2.7}$$

Now, regarding each $I \in \Phi^{(n)}$ as an element in $SL_n(\mathbb{Z})$, we have

$$\Delta_I^\circ = I\Delta_{(e_1,\ldots,e_n)}^\circ,$$

where $e_1, \ldots, e_n$ are the standard basis, i.e., $e_i = {}^t(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$, and hence

$$\begin{aligned}&\sum_{I\in\Phi^{(n)}} \int_{\Delta_I^\circ} N_{w^*}(y)^{k-1} \sum_{x\in T_{w,+}\cap\mathbb{Z}^n} \frac{1}{\langle x,y\rangle^{nk}}\omega(y)\\ &= \sum_{I\in\Phi^{(n)}} \int_{\Delta_{(e_1,\ldots,e_n)}^\circ} N_{w^*}(Iy)^{k-1} \sum_{x\in T_{w,+}\cap\mathbb{Z}^n} \frac{1}{\langle x,Iy\rangle^{nk}}\omega(Iy)\\ &= \sum_{I\in\Phi^{(n)}} \int_{\Delta_{(e_1,\ldots,e_n)}^\circ} N_{w^*}(Iy)^{k-1} \sum_{x\in {}^tIT_{w,+}\cap\mathbb{Z}^n} \frac{1}{\langle x,y\rangle^{nk}}\omega(y).\end{aligned} \tag{2.8}$$

Now, we expand $N_{w^*}(Iy)^{k-1}$ and set

$$N_{w^*}(Iy)^{k-1} =: \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^n\\ |\boldsymbol{k}|=n(k-1)}} c_{I,\boldsymbol{k}} y^{\boldsymbol{k}} \in \mathbb{Q}[y_1, \ldots, y_n]$$

for some $c_{I,\boldsymbol{k}} \in \mathbb{Q}$, i.e., $c_{I,\boldsymbol{k}}$ is the coefficient of $y^{\boldsymbol{k}} = y_1^{k_1}\cdots y_n^{k_n}$ in $N_{w^*}(Iy)^{k-1}$. Then by using Proposition 2.12 again (with $\alpha_i = e_i$), we further find

$$\begin{aligned}&\sum_{I\in\Phi^{(n)}} \int_{\Delta_{(e_1,\ldots,e_n)}^\circ} N_{w^*}(Iy)^{k-1} \sum_{x\in {}^tIT_{w,+}\cap\mathbb{Z}^n} \frac{1}{\langle x,y\rangle^{nk}}\omega(y)\\ &= \sum_{I\in\Phi^{(n)}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^n\\ |\boldsymbol{k}|=n(k-1)}} c_{I,\boldsymbol{k}} \sum_{x\in {}^tIT_{w,+}\cap\mathbb{Z}^n} \int_{\Delta_{(e_1,\ldots,e_n)}^\circ} y^{\boldsymbol{k}} \frac{1}{\langle x,y\rangle^{nk}}\omega(y)\\ &= \sum_{I\in\Phi^{(n)}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^n\\ |\boldsymbol{k}|=n(k-1)}} \frac{\boldsymbol{k}!}{(nk-1)!} c_{I,\boldsymbol{k}} \sum_{x\in {}^tIT_{w,+}\cap\mathbb{Z}^n} \frac{1}{x^{\boldsymbol{k}+\boldsymbol{1}}}\\ &= \frac{1}{(nk-1)!} \sum_{I\in\Phi^{(n)}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^n\\ |\boldsymbol{k}|=n(k-1)}} \boldsymbol{k}! c_{I,\boldsymbol{k}} \zeta_{{}^tIT_{w,+}}(\boldsymbol{k}+\boldsymbol{1}),\end{aligned} \tag{2.9}$$

where $\mathbf{1} = (1, \ldots, 1)$ and $\boldsymbol{k} + \mathbf{1} = (k_1 + 1, \ldots, k_n + 1)$ for $\boldsymbol{k} = (k_1, \ldots, k_n)$. Therefore, by combining Proposition 2.11, (2.7), (2.8), and (2.9), we obtain

$$\zeta_{F,+}(\mathfrak{a}^{-1}, k) = \frac{N\mathfrak{a}^{k-1}}{((k-1)!)^n \sqrt{d_F}} \sum_{I \in \Phi^{(n)}} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_{\geq 1}^n \\ |\boldsymbol{k}| = nk}} (\boldsymbol{k} - \mathbf{1})! c_{I, \boldsymbol{k} - \mathbf{1}} \zeta_{{}^t I T_{w,+}}(\boldsymbol{k}), \tag{2.10}$$

where $\boldsymbol{k} - \mathbf{1} = (k_1 - 1, \ldots, k_n - 1)$. Finally, for $I \in \Phi^{(n)} \subset (C_{(w^{(1)}, \ldots, w^{(n)})})^n$, by Lemma 2.6 (1), we see that ${}^t I T_{w,+}$ is a totally positive cone, and by Lemma 2.6 (2), we see that ${}^t I T_{w,+}$ is also F'-rational. This completes the proof. $\square$

3. Examples

In this section, we illustrate our main theorem with some examples.

3.1. The case of $F = \mathbb{Q}(\sqrt{5})$. In this subsection, we consider the case where $n = 2$, $F = \mathbb{Q}(\sqrt{5})$, and $\mathfrak{a} = \mathcal{O}_F = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, i.e., Example 1.4.

In this case, first we have

$$\mathcal{O}_F^\times = \{\pm 1\} \times \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^\nu \,\middle|\, \nu \in \mathbb{Z} \right\},$$
$$\mathcal{O}_{F,+}^\times = \left\{ \left(\frac{3+\sqrt{5}}{2} \right)^\nu \,\middle|\, \nu \in \mathbb{Z} \right\}$$

and hence

$$\begin{aligned} \zeta_{F,+}(\mathfrak{a}^{-1}, s) &= \sum_{x \in \mathcal{O}_{F,+}/\mathcal{O}_{F,+}^\times} \frac{1}{N_{F/\mathbb{Q}}(x)^s} \\ &= \sum_{x \in (\mathcal{O}_F - \{0\})/\mathcal{O}_F^\times} \frac{1}{|N_{F/\mathbb{Q}}(x)|^s} \\ &= \zeta_{\mathbb{Q}(\sqrt{5})}(s), \end{aligned}$$

where $\zeta_{\mathbb{Q}(\sqrt{5})}(s)$ is the Dedekind zeta function of $\mathbb{Q}(\sqrt{5})$. Moreover, we have $d_F = 5$.

Now, let us choose $w := {}^t\left(\frac{1+\sqrt{5}}{2}, 1\right)$ as a basis of $\mathfrak{a}$ over $\mathbb{Z}$. Then the dual basis $w^* = {}^t(w_1^*, w_2^*)$, the dual norm polynomial N_{w^*}, and the F-rational cone $T_{w,+}$ can be computed as follows:

$$w^* = {}^t\left(\frac{1}{\sqrt{5}}, \frac{-1+\sqrt{5}}{2\sqrt{5}} \right),$$
$$N_{w^*}(x_1, x_2) = \frac{1}{5}(-x_1^2 + x_1 x_2 + x_2^2),$$
$$\begin{aligned} T_{w,+} &= \mathbb{R}_{>0} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} + \mathbb{R}_{>0} \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} \\ &= \mathbb{R}_{>0} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} + \mathbb{R}_{>0} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}. \end{aligned}$$

Moreover, the cone decomposition Φ in the sense of Proposition 2.13 can be taken as $\Phi = \{I, J\}$ with

$$I = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad J = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Then its two-dimensional part $\Phi^{(2)}$ (cf. (2.6)) becomes $\Phi^{(2)} = \{I\}$.

Therefore, by putting

$$C := {}^tIT_{w,+} = \mathbb{R}_{>0}\begin{pmatrix} \frac{3+\sqrt{5}}{2} \\ 1 \end{pmatrix} + \mathbb{R}_{>0}\begin{pmatrix} \frac{3-\sqrt{5}}{2} \\ 1 \end{pmatrix},$$

we obtain the following from (2.10) in the proof of Theorem 2.3.

Corollary 3.1. *For $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\zeta_{\mathbb{Q}(\sqrt{5})}(k) = \frac{1}{((k-1)!)^2\sqrt{5}} \sum_{\substack{(k_1,k_2)\in\mathbb{Z}^2_{\geq 1} \\ k_1+k_2=2k}} (k_1-1)!(k_2-1)! c_{I,(k_1-1,k_2-1)} \zeta_C(k_1,k_2),$$

where $c_{I,(k_1-1,k_2-1)}$ is the coefficient of $y_1^{k_1-1}y_2^{k_2-1}$ in

$$N_{w^*}(Iy)^{k-1} = N_{w^*}(y_1, y_1+y_2)^{k-1} = \left(\frac{1}{5}(y_1^2+3y_1y_2+y_2^2)\right)^{k-1}.$$

This shows (1.1) in Example 1.4. In the cases where $k = 2, 3$, the coefficients $c_{I,(k_1-1,k_2-1)}$ can be computed by

$$\begin{aligned} N_{w^*}(Iy)^{2-1} &= \frac{1}{5}(y_1^2+3y_1y_2+y_2^2), \\ N_{w^*}(Iy)^{3-1} &= \frac{1}{25}(y_1^4+6y_1^3y_2+11y_1^2y_2^2+6y_1y_2^3+y_2^4). \end{aligned}$$

Moreover, we see that $\zeta_C(k_1,k_2) = \zeta_C(k_2,k_1)$ from a simple observation that $(x_1,x_2) \in C$ if and only if $(x_2,x_1) \in C$. Thus we find

$$\begin{aligned} \zeta_{\mathbb{Q}(\sqrt{5})}(2) &= \frac{1}{5\sqrt{5}}\left(2\zeta_C(3,1)+3\zeta_C(2,2)+2\zeta_C(1,3)\right) \\ &= \frac{1}{5\sqrt{5}}\left(4\zeta_C(3,1)+3\zeta_C(2,2)\right), \\ \zeta_{\mathbb{Q}(\sqrt{5})}(3) &= \frac{1}{4\cdot 25\sqrt{5}}(24\zeta_C(5,1)+6\cdot 6\zeta_C(4,2)+4\cdot 11\zeta_C(3,3) \\ &\qquad\qquad + 6\cdot 6\zeta_C(2,4)+24\zeta_C(1,5)) \\ &= \frac{1}{25\sqrt{5}}\left(12\zeta_C(5,1)+18\zeta_C(4,2)+11\zeta_C(3,3)\right), \end{aligned}$$

which shows (1.2) and (1.3) in Example 1.4.

3.2. **The case of $F = \mathbb{Q}(\cos(\frac{2\pi}{7}))$.** Let

$$\eta = \eta^{(1)} := 2\cos(\frac{2\pi}{7}), \quad \eta^{(2)} := 2\cos(\frac{4\pi}{7}), \quad \eta^{(3)} := 2\cos(\frac{6\pi}{7})$$

be the three roots of the cubic polynomial X^3+X^2-2X-1. Note that we have $\eta^{(2)} = \eta^2-2$, $\eta^{(3)} = -\eta^2-\eta+1$. Thus $F := \mathbb{Q}(\eta)$ is a totally real cubic field, and it is known that its ring of integers is $\mathbb{Z}[\eta]$ whose ideal class group is trivial and $d_F = 49$.

In this subsection, we consider the case where $n = 3$, $F = \mathbb{Q}(\eta)$, $\mathfrak{a} = \mathcal{O}_F = \mathbb{Z}[\eta]$. In this case, by setting

$$\varepsilon_1 := \eta^2 - 1, \quad \varepsilon_2 := \eta^2+\eta-2,$$

it is known that

$$\mathcal{O}_F^\times = \{\pm 1\} \times \varepsilon_1^{\mathbb{Z}} \times \varepsilon_2^{\mathbb{Z}},$$
$$\mathcal{O}_{F,+}^\times = \varepsilon_1^{2\mathbb{Z}} \times \varepsilon_2^{2\mathbb{Z}},$$

and hence

$$\zeta_F(\mathfrak{a}^{-1}, s) = \zeta_{\mathbb{Q}(\eta)}(s),$$

where $\zeta_{\mathbb{Q}(\eta)}(s)$ is the Dedekind zeta function of $F = \mathbb{Q}(\eta)$.

Now, let us choose $w := {}^t(\eta^2, \eta, 1)$ as a basis of $\mathfrak{a}$ over $\mathbb{Z}$. Then the dual basis $w^* = (w_1^*, w_2^*, w_3^*)$, the dual norm polynomial N_{w^*}, and the $F'(= F)$-rational cone $T_{w,+}$ can be computed as follows:

$$w^* = {}^t\left(\frac{1}{7}(2\eta^2 + \eta - 3), \frac{1}{7}(\eta^2 + 2\eta - 1), \frac{1}{7}(-3\eta^2 - \eta + 7)\right),$$

$$\begin{aligned} N_{w^*}(x_1, x_2, x_3) = \frac{1}{49}(&- x_1^3 - 2x_1^2 x_2 + 2x_1^2 x_3 + x_1 x_2^2 + 5x_1 x_2 x_3 \\ &+ x_1 x_3^2 + x_2^3 - 3x_2^2 x_3 - 4x_2 x_3^2 - x_3^3), \end{aligned}$$

$$T_{w,+} = \sum_{i=1}^{3} \mathbb{R}_{>0} w^{*(i)},$$

with

$$w^{(i)} = {}^t\left(\frac{1}{7}(2(\eta^{(i)})^2 + \eta^{(i)} - 3), \frac{1}{7}((\eta^{(i)})^2 + 2\eta^{(i)} - 1), \frac{1}{7}(-3(\eta^{(i)})^2 - \eta^{(i)} + 7)\right)$$

for $i = 1, 2, 3$.

Next, we describe the cone decomposition. Put

$$\alpha_0 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \alpha_1 := \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \alpha_2 := \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \alpha_3 := \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \alpha_4 := \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix},$$

and set

$$\begin{aligned} &I_1 := (\alpha_0, \alpha_2, \alpha_3), I_2 := (\alpha_0, \alpha_3, \alpha_4), I_3 := (\alpha_0, \alpha_4, \alpha_1), I_4 := (\alpha_4, \alpha_3, \alpha_1), \\ &I_5 := (\alpha_0, \alpha_1), I_6 := (\alpha_0, \alpha_2), I_7 := (\alpha_0, \alpha_3), I_8 := (\alpha_0, \alpha_4), \\ &I_9 := (\alpha_1, \alpha_4), I_{10} := (\alpha_3, \alpha_4), I_{11} := (\alpha_0), I_{12} := (\alpha_1). \end{aligned}$$

Note that $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are chosen so that $\alpha_1 = \rho_{w^*}(\varepsilon_1^2)\alpha_0$, $\alpha_2 = \rho_{w^*}(\varepsilon_2^2)\alpha_0$, $\alpha_3 = \rho_{w^*}(\varepsilon_1^2\varepsilon_2^2)\alpha_0$, and α_4 is an auxiliary vector to make cones smooth. Then, by using [3, Lemme 2.2] with totally positive fundamental units $\varepsilon_1^2, \varepsilon_2^2$, we find that $\Phi = \{I_i \mid i = 1, \ldots, 12\}$ gives a cone decomposition in the sense of Proposition 2.13. In this case, the three-dimensional part $\Phi^{(3)}$ becomes

$$\Phi^{(3)} = \{I_1, I_2, I_3, I_4\}.$$

Therefore, by setting

$$C_i := {}^t I_i T_{w,+}, \quad i = 1, \ldots, 4,$$

we obtain the following form (2.10).

Corollary 3.2. *For $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\zeta_{\mathbb{Q}(\eta)}(k) = \frac{1}{7((k-1)!)^3} \sum_{i=1}^{4} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^3_{\geq 1} \\ |\boldsymbol{k}|=3k}} (\boldsymbol{k}-\boldsymbol{1})! c_{I_i,\boldsymbol{k}-\boldsymbol{1}} \zeta_{C_i}(\boldsymbol{k}),$$

where $c_{I_i,\boldsymbol{k}-\boldsymbol{1}}$ is the coefficient of $y_1^{k_1-1} y_2^{k_2-1} y_3^{k_3-1}$ in $N_{w^*}(I_i y)^{k-1}$ for $\boldsymbol{k} = (k_1, k_2, k_3)$.

In the cases where $k = 2, 3$, by computing the coefficients $c_{I_i,\boldsymbol{k}-\boldsymbol{1}}$ explicitly, we find the following:

$$\zeta_{\mathbb{Q}(\eta)}(2) = \frac{1}{7^3} \sum_{i=1}^{4} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^3_{\geq 1} \\ |\boldsymbol{k}|=6}} c'_{i,\boldsymbol{k}-\boldsymbol{1}} \zeta_{C_i}(\boldsymbol{k}), \tag{3.1}$$

$$\zeta_{\mathbb{Q}(\eta)}(3) = \frac{1}{7^5} \sum_{i=1}^{4} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^3_{\geq 1} \\ |\boldsymbol{k}|=9}} c'_{i,\boldsymbol{k}-\boldsymbol{1}} \zeta_{C_i}(\boldsymbol{k}), \tag{3.2}$$

where the coefficients $c'_{i,\boldsymbol{k}-\boldsymbol{1}}$ are given in Table 1 and Table 2.

$\boldsymbol{k}$	$c'_{1,\boldsymbol{k}}$	$c'_{2,\boldsymbol{k}}$	$c'_{3,\boldsymbol{k}}$	$c'_{4,\boldsymbol{k}}$	$\boldsymbol{k}$	$c'_{1,\boldsymbol{k}}$	$c'_{2,\boldsymbol{k}}$	$c'_{3,\boldsymbol{k}}$	$c'_{4,\boldsymbol{k}}$
(4,1,1)	6	6	6	42	(2,1,3)	12	28	10	14
(3,2,1)	12	10	14	28	(1,4,1)	6	6	42	6
(3,1,2)	10	14	12	28	(1,3,2)	12	14	28	10
(2,3,1)	10	12	28	14	(1,2,3)	10	28	14	12
(2,2,2)	13	21	21	21	(1,1,4)	6	42	6	6

TABLE 1. $c'_{i,\boldsymbol{k}}$ for $k = 2$

$\boldsymbol{k}$	$c'_{1,\boldsymbol{k}}$	$c'_{2,\boldsymbol{k}}$	$c'_{3,\boldsymbol{k}}$	$c'_{4,\boldsymbol{k}}$	$\boldsymbol{k}$	$c'_{1,\boldsymbol{k}}$	$c'_{2,\boldsymbol{k}}$	$c'_{3,\boldsymbol{k}}$	$c'_{4,\boldsymbol{k}}$
(7,1,1)	90	90	90	4410	(3,1,5)	276	1764	222	462
(6,2,1)	180	150	210	2940	(2,6,1)	150	180	2940	210
(6,1,2)	150	210	180	2940	(2,5,2)	258	378	2058	336
(5,3,1)	276	222	462	1764	(2,4,3)	327	735	1281	462
(5,2,2)	258	336	378	2058	(2,3,4)	318	1302	693	504
(5,1,3)	222	462	276	1764	(2,2,5)	258	2058	336	378
(4,4,1)	279	279	945	945	(2,1,6)	180	2940	150	210
(4,3,2)	327	462	735	1281	(1,7,1)	90	90	4410	90
(4,2,3)	318	693	504	1302	(1,6,2)	180	210	2940	150
(4,1,4)	279	945	279	945	(1,5,3)	276	462	1764	222
(3,5,1)	222	276	1764	462	(1,4,4)	279	945	945	279
(3,4,2)	318	504	1302	693	(1,3,5)	222	1764	462	276
(3,3,3)	349	847	847	847	(1,2,6)	150	2940	210	180
(3,2,4)	327	1281	462	735	(1,1,7)	90	4410	90	90

TABLE 2. $c'_{i,\boldsymbol{k}}$ for $k = 3$

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